JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 73, 491–500 (1980)

# On Some New Integrodifferential Inequalities of the Wendroff Type

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This paper presents some new integrodifferential inequalities in two independent variables of the Wendroff type which can be used in the analysis of a class of hyperbolic partial integrodifferential equations as handy tools.

## 1. INTRODUCTION

In 1919, T. H. Gronwall [4] proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature. In a book published in 1961, Beckenback and Bellman [1, p. 154] stated without proof a two-independent-variable generalization of this inequality due to Wendroff which has not received the attention it deserves. Wendroff's inequality has recently evoked lively interest, as may be seen from the papers of Snow [7, 8] and Ghoshal and Masood [2, 3], which are motivated by certain applications in the theory of hyperbolic partial differential equations. The aim of the present paper is to establish some new integrodifferential inequalities in two independent variables which can be used in some problems in the theory of hyperbolic partial integrodifferential equations. The author believes that the inequalities established in this paper are new to the literature.

## 2. MAIN RESULTS

In this section we state and prove our main results on integrodifferential inequalities in two independent variables which claim the following as their origin.

LEMMA (Wendroff [1, p. 154]). *Let $\phi(x, y)$ and $c(x, y)$ be nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, for which the inequality*

$$\phi(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y c(s, t)\, \phi(s, t)\, ds\, dt \tag{1}$$



0022-247X/80/020315-23$02.00/0

*holds for $x \geqslant 0$, $y \geqslant 0$, where $a(x)$, $b(y) > 0$; $a'(x)$, $b'(y) \geqslant 0$ are continuous functions defined for $x \geqslant 0$, $y \geqslant 0$. Then*

$$\phi(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \exp\left(\int_0^x \int_0^y c(s, t)\, ds\, dt\right), \tag{2}$$

*for all $x \geqslant 0$, $y \geqslant 0$.*

The proof of this Lemma is indicated as follows.

Define a function $u(x, y)$ by the right member of (1). Then

$$u_{xy}(x, y) = c(x, y)\, \phi(x, y), \tag{3}$$

and $u(0, y) = a(0) + b(y)$, $u(x, 0) = a(x) + b(0)$. Using $\phi(x, y) \leqslant u(x, y)$ from (1) in (3) we have

$$u_{xy}(x, y) \leqslant c(x, y)\, u(x, y). \tag{4}$$

From (4) we observe that

$$\frac{u_{xy}(x, y)}{u(x, y)} \leqslant c(x, y) + \frac{u_x(x, y)\, u_y(x, y)}{u^2(x, y)},$$

i.e.,

$$\frac{\partial}{\partial y}\left(\frac{u_x(x, y)}{u(x, y)}\right) \leqslant c(x, y), \tag{5}$$

for $x \geqslant 0$, $y \geqslant 0$. Now integrating both sides of (5), first with respect to $y$ from 0 to $y$ and then with respect to $x$ from 0 to $x$, we obtain

$$u(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \exp\left(\int_0^x \int_0^y c(s, t)\, ds\, dt\right).$$

Now subsituting this value of $u(x, y)$ in (1) we obtain the desired bound in (2).

In [2, 3, 7, 8], the authors have obtained some useful generalizations of this lemma by using Riemann's functions. Our results given below are partly motivated by the fundamental integrodifferential inequality recently established by this author in [6, Theorem 1] and the integral inequality established in [5, Theorem 1].

A useful two-independent-variable integrodifferential inequality is embodied in the following theorem.

THEOREM 1. *Let $\phi(x, y)$, $\phi_{xy}(x, y)$, and $c(x, y)$ be nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, and $\phi(x, 0) = \phi(0, y) = 0$, for which the inequality*

$$\phi_{xy}(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y c(s, t)(\phi(s, t) + \phi_{st}(s, t))\, ds\, dt, \tag{6}$$

*holds for* $x \geqslant 0$, $y \geqslant 0$, *where* $a(x), b(y) > 0$; $a'(x), b'(y) \geqslant 0$ *are continuous functions defined for* $x \geqslant 0$, $y \geqslant 0$. *Then*

$$\phi_{xy}(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y c(s, t) \left[\frac{[a(0) + b(t)][a(s) + b(0)]}{[a(0) + b(0)]} \times \exp\left(\int_0^s \int_0^t [1 + c(m, n)]\, dm\, dn\right)\right] ds\, dt, \tag{7}$$

*for all* $x \geqslant 0$, $y \geqslant 0$.

*Proof.* Define a function $u(x, y)$ by

$$\begin{aligned} u(x, y) &= a(x) + b(y) + \int_0^x \int_0^y c(s, t)(\phi(s, t) + \phi_{st}(s, t))\, ds\, dt, \\ u(0, y) &= a(0) + b(y), \qquad u(x, 0) = a(x) + b(0); \end{aligned} \tag{8}$$

then (6) can be restated as

$$\phi_{xy}(x, y) \leqslant u(x, y). \tag{9}$$

Differentiating (8), first with respect to $x$ and then with respect to $y$, we have

$$u_{xy}(x, y) = c(x, y)(\phi(x, y) + \phi_{xy}(x, y)). \tag{10}$$

Integrating both sides of (9), first with respect to $y$ from 0 to $y$ and then with respect to $x$ from 0 to $x$, we have

$$\phi(x, y) \leqslant \int_0^x \int_0^y u(s, t)\, ds\, dt. \tag{11}$$

Now, using (9) and (11) in (10) we obtain

$$u_{xy}(x, y) \leqslant c(x, y)\left[u(x, y) + \int_0^x \int_0^y u(s, t)\, ds\, dt\right]. \tag{12}$$

If we put

$$\begin{aligned} v(x, y) &= u(x, y) + \int_0^x \int_0^y u(s, t)\, ds\, dt, \\ v(0, y) &= u(0, y), \qquad v(x, 0) = u(x, 0), \end{aligned} \tag{13}$$

then we have

$$v_{xy}(x, y) = u_{xy}(x, y) + u(x, y). \tag{14}$$

Using the facts that $u_{xy}(x, y) \leqslant c(x, y)\, v(x, y)$ from (12) and $u(x, y) \leqslant v(x, y)$ from (13), we see that the inequality

$$v_{xy}(x, y) \leqslant [1 + c(x, y)]\, v(x, y)$$

is satisfied, which by following an argument similar to that in the proof of lemma yields the estimate for $v(x, y)$ such that

$$v(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \exp\left(\int_0^x \int_0^y [1 + c(s, t)]\, ds\, dt\right).$$

Substituting this value of $v(x, y)$ in (12) and integrating both sides, first with respect to $y$ from 0 to $y$ and then with respect to $x$ from 0 to $x$, we have

$$u(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y c(s, t) \left[\frac{[a(0) + b(t)][b(s) + b(0)]}{[a(0) + b(0)]}\right.$$

$$\left.\times \exp\left(\int_0^s \int_0^t [1 + c(m, n)]\, dm\, dn\right)\right] ds\, dt.$$

Now substituting this value of $u(x, y)$ in (9) we obtain the desired bound in (7).

It is interesting to note that the advantage of inequality (7) over inequality (6) becomes apparent if we consider $a(x)$, $b(y)$, $c(x, y)$ as known functions and $\phi(x, y)$ and $\phi_{xy}(x, y)$ as unknown; that is, inequality (7) gives us a completely known function on the right side which majorizes $\phi_{xy}(x, y)$ and consequently $\phi(x, y)$ after integration, first with respect to $y$ from 0 to $y$ and then with respect to $x$ from 0 to $x$.

In the special case when $a(x) + b(y) = k$, for $x \geqslant 0$, $y \geqslant 0$, where $k > 0$ is a constant, the bound obtained in (7) reduces to

$$\phi_{xy}(x, y) \leqslant k \left[1 + \int_0^x \int_0^y c(s, t) \exp\left(\int_0^s \int_0^t [1 + c(m, n)]\, dm\, dn\right) ds\, dt\right],$$

which can be used in some applications.

Another interesting and useful integrodifferential inequality in two independent variables may be stated as follows.

THEOREM 2. *Let $\phi(x, y)$, $\phi_x(x, y)$, $\phi_y(x, y)$, $\phi_{xy}(x, y)$, and $c(x, y)$ be nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, and $\phi(x, 0) = \phi(0, y) = 0$, for which the inequality*

$$\phi_{xy}(x, y) \leqslant a(x) + b(y) + M \left[\phi(x, y) + \int_0^x \int_0^y c(s, t)(\phi(s, t) + \phi_{st}(s, t))\, ds\, dt\right], \tag{15}$$

*holds for $x \geqslant 0$, $y \geqslant 0$, where $a(x)$, $b(y) > 0$; $a'(x)$, $b'(y) \geqslant 0$ are continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, and $M \geqslant 0$ is a constant. Then*

$$\phi_{xy}(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \times \exp\left(\int_0^x \int_0^y [M + c(s, t) + Mc(s, t)]\, ds\, dt\right) \tag{16}$$

*for all $x \geqslant 0$, $y \geqslant 0$.*

*Proof.* Define

$$u(x, y) = a(x) + b(y) + M\left[\phi(x, y) + \int_0^x \int_0^y c(s, t)(\phi(s, t) + \phi_{st}(s, t))\, ds\, dt\right],$$
$$u(0, y) = a(0) + b(y), \qquad u(x, 0) = a(x) + b(0). \tag{17}$$

Differentiating (17) first with respect to $x$ and then with respect to $y$ we have

$$u_{xy}(x, y) = M[\phi_{xy}(x, y) + c(x, y)(\phi(x, y) + \phi_{xy}(x, y))]. \tag{18}$$

Using the facts that $\phi_{xy}(x, y) \leqslant u(x, y)$ from (15) and $M\phi(x, y) \leqslant u(x, y)$ from (17) in (18) we see that the inequality

$$u_{xy}(x, y) \leqslant [M + c(x, y) + Mc(x, y)]\, u(x, y)$$

is satisfied, which implies the estimation for $u(x, y)$ such that

$$u(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \times \exp\left(\int_0^x \int_0^y [M + c(s, t) + Mc(s, t)]\, ds\, dt\right).$$

Now, substituting this value of $u(x, y)$ in (15) we obtain the desired bound in (16).

We note that in the special case when $a(x) + b(y) = k$, for $x \geqslant 0$, $y \geqslant 0$, where $k > 0$ is a constant, the bound obtained in (16) reduces to

$$\phi_{xy}(x, y) \leqslant k \exp\left(\int_0^x \int_0^y [M + c(s, t) + Mc(s, t)]\, ds\, dt\right),$$

which may be used in certain situations.

Further, we note that with suitable alterations in Theorem 2 we can obtain a bound for the integrodifferential inequality in two independent variables of the form

$$\phi_{xy}(x, y) \leqslant a(x) + b(y) + M\left[\phi(x, y) + \int_0^x \int_0^y c(s, t)\, \phi_{st}(s, t)\, ds\, dt\right]$$

such that

$$\phi_{xy}(x, y) \leqslant \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \exp\left(\int_0^x \int_0^y M[1 + c(s, t)]\, ds\, dt\right),$$

which may be used conveniently in some applications.

In concluding this section we establish the following two-independent-variable generalization of the integral inequality recently established by the present author in [5, Theorem 1].

THEOREM 3. *Let $\phi(x, y)$, $p(x, y)$, and $q(x, y)$ be nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$, for which the inequality*

$$\phi(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y p(s, t)\, \phi(s, t)\, ds\, dt$$
$$+ \int_0^x \int_0^y p(s, t) \left(\int_0^s \int_0^t q(m, n)\, \phi(m, n)\, dm\, dn\right) ds\, dt, \tag{19}$$

*holds for $x \geqslant 0$, $y \geqslant 0$, where $a(x)$, $b(y) > 0$; $a'(x)$, $b'(y) \geqslant 0$ are continuous functions defined for $x \geqslant 0$, $y \geqslant 0$. Then*

$$\phi(x, y) \leqslant a(x) + b(y) + \int_0^x \int_0^y p(s, t)\, Q(s, t)\, ds\, dt \tag{20}$$

*for all $x \geqslant 0$, $y \geqslant 0$, where*

$$Q(x, y) = \frac{[a(0) + b(y)][a(x) + b(0)]}{[a(0) + b(0)]} \exp\left(\int_0^x \int_0^y [p(s, t) + q(s, t)]\, ds\, dt\right), \tag{21}$$

*for $x \geqslant 0$, $y \geqslant 0$.*

The proof of this theorem follows by an argument similar to that in the proof of the one-variable case (see [5, Theorem 1]) in view of the proof of Theorem 1 given above. We omit the details.

In the particular case when $a(x) + b(y) = k$, for $x \geqslant 0$, $y \geqslant 0$, where $k > 0$ is a constant, then the bound obtained in (20) reduces to

$$\phi(x, y) \leqslant k\left[1 + \int_0^x \int_0^y p(s, t) \exp\left(\int_0^s \int_0^t [p(m, n) + q(m, n)]\, dm\, dn\right) ds\, dt\right]$$

for all $x \geqslant 0$, $y \geqslant 0$.

## 3. Some Applications

In this section we present some applications of our results to the study of boundedness and continuous dependence of the solutions of some partial integrodifferential equations. There are many possible applications of the inequalities established in this paper, but those presented here are sufficient to convey the importance of our results.

Example 1. As a first application, consider the nonlinear hyperbolic partial integrodifferential equation

$$u_{xy}(x, y) = f(x, y) + \int_0^x \int_0^y F(x, y, s, t, u(s, t), u_{st}(s, t))\, ds\, dt \tag{22}$$

with the boundary conditions $u(0, y) = u(x, 0) = 0$, where $f$ and $F$ are continuous functions such that

$$|f(x, y)| \leqslant M,$$

and

$$|F(x, y, s, t, u(s, t), u_{st}(s, t))| \leqslant c(s, t)[|u(s, t)| + |u_{st}(s, t)|]$$

for $x \geqslant 0, y \geqslant 0$, where $M > 0$ is a constant and $c(x, y)$ is a continuous function defined for $x \geqslant 0$, $y \geqslant 0$. If $\phi(x, y)$ be any solution of this boundary-value problem such that $v(x, y) = |\varphi(x, y)|$ and $v_{xy}(x, y) = |\phi_{xy}(x, y)|$, then

$$\phi_{xy}(x, y) = f(x, y) + \int_0^x \int_0^y F(x, y, s, t, \phi(s, t), \phi_{st}(s, t))\, ds\, dt.$$

Hence for $x \geqslant 0$, $y \geqslant 0$, we have

$$|\phi_{xy}(x, y)| \leqslant M + \int_0^x \int_0^y c(s, t)[|\phi(s, t)| + |\phi_{st}(s, t)|]\, ds\, dt.$$

Now by a suitable application of Theorem 1 we have

$$|\phi_{xy}(x, y)| \leqslant M \left[1 + \int_0^x \int_0^y c(s, t) \exp\left(\int_0^s \int_0^t [1 + c(m, n)]\, dm\, dn\right) ds\, dt\right],$$

for $x \geqslant 0$, $y \geqslant 0$. Further integrating both sides of the above inequality first with respect $y$ from 0 to $y$ and then with respect to $x$ from 0 to $x$ we obtain the bound on the solution $\phi(x, y)$ of (1).

EXAMPLE 2. Our second application is an example of continuous dependence of the solution on the equation and boundary data. Consider the two boundary-value problems

$$u_{xy}(x, y) = f\left[x, y, u(x, y), \int_{x_0}^{x}\int_{y_0}^{y} k(x, y, s, t, u(s, t))\, ds\, dt\right],$$

$$u(x_0, y) = g(y), \qquad u(x, y_0) = h(x), \qquad g(y_0) = h(x_0) \tag{23}$$

and

$$U_{xy}(x, y) = F\left[x, y, U(x, y), \int_{x_0}^{x}\int_{y_0}^{y} k_0(x, y, s, t, U(s, t))\, ds\, dt\right],$$

$$U(x_0, y) = G(y), \qquad U(x, y_0) = H(x), \qquad G(y_0) = H(x_0), \tag{24}$$

where all functions are continuous and $f$ and $k$ in (23) satisfy

$$| f[x, y, u, v] - f[x, y, \bar{u}, \bar{v}]| \leqslant M[| u - \bar{u} | + | v - \bar{v} |],$$

and

$$| k(x, y, s, t, u) - k(x, y, s, t, \bar{u})| \leqslant q(s, t) \mid u - \bar{u} \mid,$$

where $M > 0$ is a constant and $q(s, t) \geqslant 0$ is a continuous function defined for $s \geqslant 0$, $t \geqslant 0$. The equivalent integral equations are

$$\begin{aligned} u(x, y) &= g(y) + h(x) - g(y_0) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} f\left[s, t, u(s, t), \int_{x_0}^{s}\int_{y_0}^{t} k(s, t, m, n, u(m, n))\, dm\, dn\right] ds\, dt, \end{aligned}$$

and

$$\begin{aligned} U(x, y) &= G(y) + H(x) - G(y_0) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} F\left[s, t, U(s, t), \int_{x_0}^{s}\int_{y_0}^{t} k_0(s, t, m, n, U(m, n))\, dm\, dn\right] ds\, dt. \end{aligned}$$

Then

$$\begin{aligned} u - U &= (g - G) + (h - H) - [g(y_0) - G(y_0)] \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} \left\{ f\left[s, t, u, \int_{x_0}^{s}\int_{y_0}^{t} k(s, t, m, n, u)\, dm\, dn\right] \right. \\ &\quad \left. - F\left[s, t, U, \int_{x_0}^{s}\int_{y_0}^{t} k_0(s, t, m, n, U)\, dm\, dn\right] \right\} ds\, dt \end{aligned}$$

By adding and subtracting $f[s, t, U, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, m, n, U)\, dm\, dn]$ in the integrand we obtain

$$\begin{aligned} | u - U | \leqslant{} & | g - G | + | h - H | + | g(y_0) - G(y_0)| \\ & + \int_{x_0}^{x} \int_{y_0}^{y} \Big| f\Big[s, t, u, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, m, n, u)\, dm\, dn\Big] \\ & - f\Big[s, t, U, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, m, n, U)\, dm\, dn\Big] \Big|\, ds\, dt \\ & + \int_{x_0}^{x} \int_{y_0}^{y} \Big| f\Big[s, t, U, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, m, n, U)\, dm\, dn\Big] \\ & - F\Big[s, t, U, \int_{x_0}^{s} \int_{y_0}^{t} k_0(s, t, m, n, U)\, dm\, dn\Big]\Big|\, ds\, dt, \end{aligned}$$

for $x \geqslant 0$, $y \geqslant 0$. If

$$| g - G | \leqslant \epsilon, \qquad | h - H | \leqslant \epsilon,$$

and

$$\begin{aligned} & \int_{x_0}^{x} \int_{y_0}^{y} \Big| f\Big[s, t, u, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, m, n, u)\, dm\, dn\Big] \\ & \quad - F\Big[s, t, U, \int_{x_0}^{s} \int_{y_0}^{t} k_0(s, t, m, n, U)\, dm\, dn\Big]\Big|\, ds\, dt \leqslant \epsilon, \end{aligned}$$

then

$$\begin{aligned} & | u - U | \\ & \quad \leqslant 3\epsilon + \epsilon + \int_{x_0}^{x} \int_{y_0}^{y} M \Big[| u - U | + \int_{x_0}^{s} \int_{y_0}^{t} q(m, n)\, | u - U |\, dm\, dn\Big]\, ds\, dt. \end{aligned}$$

By a suitable application of Theorem 3 we have

$$| u - U | \leqslant 4\epsilon \Big\{1 + \int_{x_0}^{x} \int_{y_0}^{y} M \exp\Big(\int_{x_0}^{s} \int_{y_0}^{t} [M + q(m, n)]\, dm\, dn\Big)\, ds\, dt\Big\},$$

for $x \geqslant 0$, $y \geqslant 0$. On the compact set, if the quantity in braces is bounded by some constant $N$, then $| u - U | \leqslant N\epsilon$ on this set, so the solution to such a boundary-value problem depends continuously on $f$ and the boundary values. If $\epsilon \to 0$, then $| u - U | \to 0$ on the set.

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